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# Generalized Koszul duality and its application

(joint work with A. Takahashi)

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## 0. Motivation

Let  $k$  be a field and  $V$  be a finite dimensional vector space over  $k$ . Then we have the following famous diagram.

$$\begin{array}{ccc} \mathcal{D}^b(\text{coh } \mathbb{P}(V)) & \xrightarrow[\sim]{\text{Beilinson}} & \mathcal{D}^b(\text{mod-} B_n) \\ \text{BGG corresp.} \downarrow & & \\ \text{grmod } \bigwedge(V^*) & & \end{array}$$

where

$$\begin{cases} B_n : (\text{the Beilinson algebra}) \\ \bigwedge(V^{**}) : (\text{the exterior algebra of } V^*) \\ \text{grmod} : (\text{the stable category}) \end{cases}$$

The BGG correspondence was developed into the theory of Koszul duality.

Let  $Q$  be a quiver whose path algebra  $kQ$  is finite dimensional and has infinite representation type.

Then we have the following diagram similar to the above diagram.

$$\begin{array}{ccc} \mathcal{D}^b(\text{qcoh } \Pi(Q)) & \xleftarrow[\sim]{M.} & \mathcal{D}^b(\text{mod-} kQ) \\ & \searrow \text{Happel} & \\ \text{grmod } T(Q) & & \end{array}$$

where

$$\begin{cases} D(kQ) := \text{Hom}_{k\text{-vect}}(kQ, k) \\ \rho := \mathbb{R} \text{Hom}_{kQ}(D(kQ), kQ)[1] \\ \Pi(Q) := \bigoplus_{n \geq 0} \rho^{\otimes n} \quad (\text{the preprojective algebra}) \\ T(Q) := kQ \oplus D(kQ) \quad (\text{the trivial extension algebra}) \end{cases}$$

Comparing these diagrams, it is natural to consider that  $T(Q)$  and  $\Pi(Q)$  are “Koszul dual” to each other. But they are not Koszul algebra in the classical sense. To get a framework including this case, we work with graded dg-algebras.

## 1. Generalized Koszul duality

Let  $\mathcal{A}$  be a dg-algebra over  $k$  and

$$\mathcal{R} = \mathcal{A} \oplus P_1 \langle -1 \rangle \oplus P_2 \langle -2 \rangle \oplus P_3 \langle -3 \rangle \oplus \dots$$

be a connected positively graded dg-algebra over  $\mathcal{A}$ . (where  $\langle n \rangle$  is the graded-degree shift operator by  $n$ .) We assume that each term  $P_i$  is obtained from  $\mathcal{A} \otimes_k \mathcal{A}^{op}$  by taking finite number of cones, shifts and direct summand.

We construct its “Koszul dual”  $\mathcal{R}^!$ .

**Definition 1.**

$$Q_n := \left( \bigoplus_{\lambda \models n} \bigotimes_{\mathcal{A}} (P_{\lambda_i}[-\lambda_i + 1]), \text{“one-sided-twist”} \right)$$

where the “one-sided-twist” is obtained from the totalization of the bar resolution of  $\mathcal{R}$  over  $\mathcal{A}$ .

$$\mathcal{R}^! := \mathcal{A} \oplus Q_1^\vee[-1]\langle 1 \rangle \oplus Q_2^\vee[-2]\langle 2 \rangle \oplus Q_3^\vee[-3]\langle 3 \rangle \oplus \dots$$

where we set  $(-)^\vee := \text{Hom}_{\mathcal{A}\text{-bimod}}(-, \mathcal{A})$ .

This  $\mathcal{R}^!$  is a connected negatively graded dg-algebra over  $\mathcal{A}$ .  $\mathcal{R}$  and  $\mathcal{R}^!$  satisfy the following properties similar to the classical case

**Theorem 2.**

$$\mathbb{R} \text{Hom}_{\mathcal{R}\text{-gr}}(\mathcal{A}, \mathcal{A}) \simeq \mathcal{R}^!$$

$$\mathbb{R} \text{Hom}_{\mathcal{R}^!\text{-gr}}(\mathcal{A}, \mathcal{A}) \simeq \mathcal{R}.$$

We can also construct the “Koszul complex”  $\mathcal{K}$  which is graded  $\mathcal{R} \otimes (\mathcal{R}^!)^{op}$  module. For any graded  $\mathcal{R}$  module  $\mathcal{M}$ , we have

$$\text{Hom}_{\text{gr-}\mathcal{R}}(\mathcal{K}, \mathcal{M}) \simeq \mathbb{R} \text{Hom}_{\text{gr-}\mathcal{R}}(\mathcal{A}, \mathcal{M}).$$

From the functor

$$F := \text{Hom}_{\text{gr-}\mathcal{R}}(\mathcal{K}, -) : \mathcal{D}(\text{gr-}\mathcal{R}) \longrightarrow \mathcal{D}(\text{gr-}\mathcal{R}^!)$$

we obtain the following equivalence

**Theorem 3.**

$$\text{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}) \simeq \text{Perf}(\text{gr-}\mathcal{R}^!) \quad (1)$$

where  $\text{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R})$  (resp.  $\text{Perf}(\text{gr-}\mathcal{R}^!)$ ) is the full sub triangulated category of  $\mathcal{D}(\text{gr-}\mathcal{R})$  (resp.  $\mathcal{D}(\text{gr-}\mathcal{R}^!)$ ) generated by  $\{\mathcal{A}\langle n \rangle \mid n \in \mathbb{Z}\}$  (resp.  $\{\mathcal{R}^!\langle n \rangle \mid n \in \mathbb{Z}\}$ ).

## 2. Application

**2-1** Let  $\mathcal{A}$  be a homologically smooth dg-algebra and  $P$  be an invertible  $\mathcal{A}$  bi-module and  $R = T_{\mathcal{A}}(P)$  be the tensor algebra of  $P$  over  $\mathcal{A}$ .

**Theorem 4.** we have the following equivalences by the functor  $F$

$$\text{Perf}(\text{gr-}\mathcal{R}) \simeq \text{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}^!) \quad (2)$$

From the equivalences 1 and 2, we get the following corollary which generalize the Happel’s theorem.

**Corollary 5.**

$$\begin{aligned} \text{Perf}(\text{gr-}\mathcal{R}) / \text{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}) &\simeq \text{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}^!) / \text{Perf}(\text{gr-}\mathcal{R}^!) \\ &\simeq \text{Perf}(\mathcal{A}) \end{aligned}$$

**2-2** Let  $A$  be a finite dimensional algebra of finite global dimension and  $P$  be a tilting  $A$  bi-module.

Reversing Serre’s vanishing theorem, we define a pair  $(\mathcal{D}^{P, \geq 0}, \mathcal{D}^{P, \leq 0})$  of full sub categories of  $\mathcal{D}^b(\text{mod-} A)$  as follows.

**Definition 6.**

$$M \in \mathcal{D}^{P, \geq 0} \iff M \otimes^L P^{\otimes n} \in \mathcal{D}^{\geq 0}(\text{mod-} A)$$

$$\begin{aligned} \text{resp. } M \in \mathcal{D}^{P, \leq 0} &\iff M \otimes^L P^{\otimes n} \in \mathcal{D}^{\leq 0}(\text{mod-} A) \\ &\text{for } n \gg 0 \end{aligned}$$

**Corollary 7.** Assume that  $P^{\otimes n}$  is pure for every  $n \geq 0$ , then the following conditions are equivalent

- (i)  $(\mathcal{D}^{P, \geq 0}, \mathcal{D}^{P, \leq 0})$  is a  $t$ -structure in  $\mathcal{D}^b(\text{mod-} A)$
- (ii) the tensor algebra  $T_A(P)$  is a graded coherent ring.